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GEOMETRY: EXAMPLES 4

1. Fix a compact abstract surface Σ. In lectures we defined the Euler characteristic of a triangulation of Σ, but one can make the same definition for an arbitrary polygonal decomposition, defined in the same way as a triangulation but with arbitrary polygons in place of triangles.
   (a) Given a polygonal decomposition, show that there exists a triangulation with the same Euler characteristic.
   (b) Consider a polygonal decomposition of Σ such that at least three edges meet at each vertex and each face has at least three edges. Let F_n denote the number of faces with exactly n edges, and V_m the number of vertices at which exactly m edges meet. Show that
   $$\sum_n nF_n = 2E = \sum_m mV_m.$$
   Deduce that if $V_3 = 0$ then $E \geq 2V$, whilst if $F_3 = 0$ then $E \geq 2F$. Conclude that if $\Sigma = S^2$ then $V_3 + F_3 > 0$.
   (c) The surface of a football is decomposed into hexagons and pentagons, with precisely 3 faces meeting at each vertex. How many pentagons are there?

2. Show that if the surfaces Σ_{g_1} and Σ_{g_2} of genus g_1 and g_2 are diffeomorphic then $g_1 = g_2$. You may assume that every smooth surface admits a Riemannian metric.

3. Let z_1, z_2 be distinct points in the hyperbolic plane, working in either the disc model or the upper half-plane model. Suppose the hyperbolic line through z_1 and z_2 meets the boundary at points w_1 and w_2, where z_1 lies between w_1 and z_2. Show that the hyperbolic distance $d(z_1, z_2)$ is equal to $\log r$, where r is the cross-ratio of the four points z_1, z_2, w_1, w_2 taken in an appropriate order.

4. Given a Riemannian surface (Σ, g), with associated distance function
   $$d(p,q) = \inf\{\text{Length}(\gamma) : \gamma \text{ a smooth path from } p \text{ to } q\},$$
   the *circle of centre p and radius r*, denoted $C(p,r)$, is the set $\{x \in \Sigma : d(x,p) = r\}$.
   (a) For the sphere (S^2, g_{round}), and for $r \in (0, \pi)$, show that each circle $C(p,r)$ has circumference $2\pi \sin r$ and bounds a disc of area $2\pi(1 - \cos r)$.
   (b) For the Poincaré disc model (D, g_D), and $r \in (0, \infty)$, show that each circle $C(p,r)$ is a Euclidean circle and that with respect to g_D it has circumference $2\pi \sinh r$ bounds a disc of area $2\pi(\cosh r - 1)$.
   (c) Show that no hyperbolic triangle contains a half-disc of (hyperbolic) radius $\geq \operatorname{arcosh} 2$. Deduce that there exists $\delta > 0$ such that in any hyperbolic triangle the union of the δ-neighbourhoods of two of the sides completely contains the third side.

5. Show that two hyperbolic lines have a common perpendicular if and only if they are ultraparallel. In this case use the Gauss–Bonnet theorem to show that the common perpendicular is unique.

6. (a) Show that the geodesic 2-gon on S^2 bounded by two great circles meeting at angle α has area 2α. Deduce the Gauss–Bonnet theorem for spherical triangles (i.e. geodesic triangles on S^2).
   (b) Show that the integral of $\frac{dx}{y}$ along a geodesic in the upper half-plane model is equal to the angle through which the geodesic turns relative to the vertical. By applying Green's theorem, deduce the Gauss–Bonnet theorem for hyperbolic polygons.

7. (a) Let φ be an orientation-preserving isometry of the hyperbolic plane not equal to the identity. Show that φ is elliptic iff in the disc model it's conjugate to a rotation. Show that φ is parabolic or hyperbolic iff in the upper half-plane model it's conjugate to $z \mapsto z \pm 1$ or a map of the form $z \mapsto az$ for $a \in (1, \infty)$ respectively.
   (b) Show that the map T from the Möbius group to $\mathbb{C}$ given by
   $$T : \left(z \mapsto \frac{az+b}{cz+d}\right) \mapsto \frac{(a+d)^2}{ad-bc}$$
   is well-defined and conjugation-invariant.

(c) Show that if a non-identity Möbius map φ sends the half-plane or disc to itself then $T(\varphi)$ is real, and that φ is elliptic, parabolic, or hyperbolic iff $T(\varphi)$ is < 4, $= 4$, or > 4 respectively.

8. Equip the cylinder with the complete hyperbolic metric obtained by quotienting the upper half-plane by the group generated by $z \mapsto az$ for $a \in (1, \infty)$. Show that this contains a unique closed geodesic, and find its length. Draw a diagram of the cylinder showing a selection of geodesics.

9. Construct a compact hyperbolic surface Σ, and disjoint simple closed geodesics $\gamma_i \subset \Sigma$, for which $\gamma_1 \sqcup \gamma_2$ bounds a subsurface Σ' of Σ homeomorphic to the complement of two disjoint discs in a torus. (Here *simple* means that each geodesic does not cross itself; formally, its image is homeomorphic to S^1.) Can this happen if Σ has genus two?